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# Local Communication in Repeated Games with Local Monitoring\*

M. Laclau<sup>†</sup>

VERY PRELIMINARY AND INCOMPLETE

## Abstract

I consider repeated games with local monitoring: each player observes his neighbors' moves only. Hence, monitoring is private and imperfect. I assume local and public communication: communication is restricted to neighbors, and each player sends the same message to each of his neighbors at each stage. Both communication and monitoring structures are given by the network. The solution concept is perfect Bayesian equilibrium. In the four-player case, a folk theorem holds if and only if the network is 2-connected. Some examples are given for games with more than four players.

**Keywords:** communication, folk theorem, imperfect private monitoring, networks, repeated games.

**JEL codes:** C72, C73

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# 1 Introduction

Many papers on folk theorems with imperfect monitoring consider global communication (Ben-Porath and Kahneman, [3]; Compte, [4]; Kandori and Matsushima, [11]; Obara, [18]; etc): players can send messages to all their opponents. However, it seems natural to consider other kinds of communication. In this paper, I study a model where communication is *local*: each player can communicate only with a subset of players, called the neighbors.

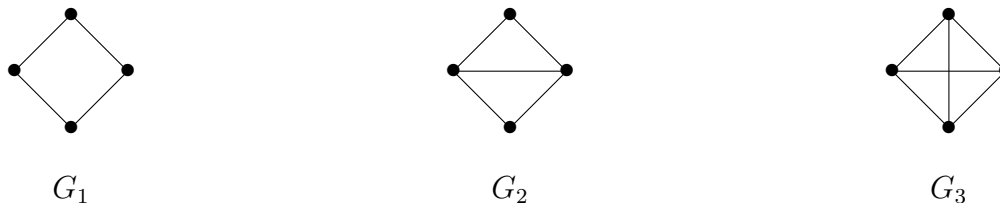
I study this model of communication in the context of repeated games with local monitoring: each player observes his neighbors' moves only. This is modeled by a network: nodes represent players, edges link neighbors. The monitoring structure is represented by the network: monitoring is private and local. In addition, players can send costless messages at each stage. This paper addresses the following question: for which networks does a *full* folk theorem hold, *i.e.* under which conditions are *all* feasible, strictly individually rational payoffs are equilibrium payoffs in the repeated game with low discounting?

I study local and public communication: at each stage, each player can send costless messages only to his neighbors (*i.e.* communication is local), and each player is constrained to send the same message to all his neighbors (*i.e.* communication is public). Hence, communication is restricted by the network structure. The solution concept is perfect Bayesian equilibrium. In the four-player case, the main result is that a folk theorem holds if and only if the network is 2-connected. With more than four players, I exhibit some preliminary results and examples.

**Application** An application of interest is a partnership game (see [19]). Consider a partnership in which production is conducted by a team whose goal is to maintain a certain level of individual effort by its members. Each member's effort is observable only by his direct colleagues (called the neighbors), and there is moral hazard (effort is costly). If a member is reported by his direct colleagues, the group can punish him by reducing his share in the total profit, which raises other members' shares.

Agents may not be able to communicate with all the members in the group. If agents can communicate publicly but only with their direct colleagues, I prove in the four-player case (Theorem 3.1) that coordination can be sustained if and only if the network is 2-connected, that is if the team's structure takes one of the following forms (see Section 3.1):

Figure 1:



**Related literature.** The folk theorem was originally established for Nash equilibria ([2, 8, 23, 24]), and extended by Fudenberg and Maskin ([8, 9]) to subgame-perfect equilibria. A key assumption is perfect monitoring. Lots of papers on folk theorems with imperfect monitoring have focused on imperfect public monitoring (see [7]). The model of collusion by Green and Porter ([10]) also considers imperfect public monitoring in that the market price serves as a commonly observable signal. In the undiscounted case, Lehrer (see [15, 16]) provides a fairly comprehensive study of the equilibrium payoff set for two-player games with imperfect private monitoring. With more than two players, the difficulty relies on the necessity for the players to coordinate their behavior to punish potential deviators. Under discounting, as assumed here, much less is known. There is a large recent literature on imperfect private monitoring and belief-free equilibria (see [17] for a general survey). Fudenberg and Levine ([6]) establish a folk theorem with imperfect private monitoring without explicit communication. They consider private random signals induced by the action profile of all other players.

With public and global communication, Compte ([4]), Kandori and Matsushima ([11]), and Obara ([18]), provide sufficient conditions for a folk theorem to hold. Closer to my setting, Ben-Porath and Kahneman ([3]) establish a folk theorem for the case in which each player observes his neighbors' moves. However, they maintain the assumption of public and global communication.

With public and local communication, Renault and Tomala ([20]), and Tomala ([25]), study repeated games with the same signals as here (*i.e.* each player observes his neighbors' moves). Yet, they do not impose sequential rationality. In [21] and [22], Renault and Tomala study local communication. The aim of both papers is to transmit information along a network (with a single possible deviator in [21], extended to  $k$  deviators in [22]). Their techniques prove useful for the case of public and local communication studied here. However, they do not apply directly here, since they assume that the set of deviators is fixed through time. On the contrary, since I impose sequential rationality here, all players may deviate unilaterally.

Finally, there is a vast literature on information transmission in networks which consider *broadcast* (*i.e.* public) or *unicast* (*i.e.* private) communication along networks (see for in-

stance [5]). However, with very few exceptions (as [22] mentioned above), both rationality and equilibrium are ignored. Deviant agents are not supposed to respond to incentives. The focus is on properties of communication protocols, such as reliability, security, etc.

The paper is organized as follows. The setup is introduced in Section 2. Section 3 introduces the main results. Finally, I raise open questions in Section 4.

## 2 The setup

### 2.1 Preliminaries

Consider a repeated game described by the following data:

- a finite set  $N = \{1, \dots, n\}$  of players ( $n \geq 2$ );
- For each player  $i \in N$ , a non-empty finite set  $A^i$  of actions (with  $\#A^i \geq 2$ ). Denote  $A = \prod_{i \in N} A^i$ .
- An undirected graph  $G = (N, E)$  in which the nodes are the players and  $E \subseteq N \times N$  is a set of links. Let  $\mathcal{N}(i) = \{j \neq i : ij \in E\}$  be the set of neighbors of player  $i$ . Since  $G$  is undirected, the following holds:  $i \in \mathcal{N}(j) \Leftrightarrow j \in \mathcal{N}(i)$ .
- A payoff function for each player  $i$  in  $N$ :  $g^i : \prod_{j \in N} A^j \rightarrow \mathbb{R}$ .
- Finally, a non-empty finite set  $M^i$  of player  $i$ 's messages. The specification of the set  $M^i$  is described in the following sections. In particular,  $M^i$  may be finite or infinite.

I use the following notations:  $A^{\mathcal{N}(i)} = \prod_{j \in \mathcal{N}(i)} A^j$ ,  $N^{-i} = N \setminus \{i\}$ , and  $g = (g^1, \dots, g^n)$  denote the payoff vector. The repeated game unfolds as follows. At every stage  $t \in \mathbb{N}^*$ :

- (i) simultaneously, players choose actions in their action sets, and send costless messages to their neighbors only. Communication is public in that each player sends the same message to each neighbor. For each player  $i$  in  $N$ , let  $m_t^i$  be player  $i$ 's message (to his neighbors) at stage  $t$ .
- (ii) Let  $a_t = (a_t^i)$  be the action profile at stage  $t$ . At the end of stage  $t$ , each player  $i \in N$  observes his neighbors' moves  $(a^j)_{j \in \mathcal{N}(i)}$ .

Hence, the monitoring and communication possibilities are given by the network  $G$ . In addition, I assume perfect recall, and that the whole description of the game is common knowledge. For each stage  $t$ , denote by  $H_t^i$  the set of private histories of player  $i$  up to stage  $t$ , that is  $H_t^i = (A^i \times (M^i)^{\mathcal{N}^{-i}} \times (M^j)_{j \in \mathcal{N}^{-i}} \times \{g^i\} \times A^{\mathcal{N}(i)})^t$ , where  $\{g^i\}$  is the range of  $g^i$

( $H_0^i$  is a singleton). An element of  $h_t^i$  is called an  $i$ -history of length  $t$ . An *action strategy* for player  $i$  is denoted by  $\sigma^i = (\sigma_t^i)_{t \geq 1}$  where for each stage  $t$ ,  $\sigma_t^i$  is a mapping from  $H_{t-1}^i$  to  $\Delta(A^i)$  (where  $\Delta(A^i)$  denotes the set of probability distributions over  $A^i$ ). A *communication strategy* for player  $i$  is denoted by  $\phi^i = (\phi_t^i)_{t \geq 1}$  where for each stage  $t$ ,  $\phi_t^i$  is a mapping from  $H_{t-1}^i$  to  $\Delta((M^i)^{\mathcal{N}^{-i}})$ . Each player can deviate from  $\sigma^i$  or from  $\phi^i$ , henceforth I shall distinguish between action and communication deviations accordingly. I call a *behavior strategy* of a player  $i$  the pair  $(\sigma^i, \phi^i)$ . Let  $\Sigma^i$  be the set of action strategies of player  $i$  and  $\Phi^i$  his set of communication strategies. I denote by  $\sigma = (\sigma^i)_{i \in N} \in \prod_{i \in N} \Sigma^i$  the joint action strategy of the players and by  $\phi = (\phi^i)_{i \in N} \in \prod_{i \in N} \Phi^i$  their joint communication strategy. Let  $H_t$  be the set of histories of length  $t$  that consists of the sequences of actions, payoffs and messages for  $t$  stages. Let  $H_\infty$  be the set of all possible infinite histories. A profile  $(\sigma, \phi)$  defines a probability distribution,  $\mathbb{P}_{\sigma, \phi}$ , over the set of plays  $H_\infty$ , and I denote  $\mathbb{E}_{\sigma, \phi}$  the corresponding expectation. I study the discounted infinitely repeated game, in which the overall payoff function of each player  $i$  in  $N$  is the expected sum of discounted payoffs. That is, for each player  $i$  in  $N$ :

$$\gamma_\delta^i(\sigma, \phi) = \mathbb{E}_{\sigma, \phi} \left[ (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} g_t^i(a_t^i, a_t^{\mathcal{N}^{(i)}}) \right],$$

where  $\delta \in [0, 1)$  is a common discount factor.

Depending on the model considered, I will study belief-free equilibria or perfect Bayesian equilibria (henceforth PBE) of the discounted repeated game (see definitions in Section ??). Let  $\Gamma_\delta(G, g)$  be the  $\delta$ -discounted game, and  $E_\delta^B(G, g)$  (respectively  $E_\delta^{PBE}(G, g)$ ) its associated set of belief-free equilibrium payoffs (respectively PBE payoffs). For each  $a \in A$ , let  $g(a) = (g^1(a), \dots, g^n(a))$ , and  $g(A) = \{g(a) : a \in A\}$ . The set of feasible payoffs is the convex hull of  $g(A)$ , denoted by  $\text{co } g(A)$  be the convex hull of  $g(A)$ . The (independent) minmax level of player  $i$  is defined by:

$$\underline{v}^i = \min_{(x^j)_{j \in N-i} \in \prod_{j \in N-i} \Delta(A^j)} \max_{x^i \in \Delta(A^i)} g^i(x^i, (x^j)_{j \in N-i}).$$

Henceforth, I shall normalize the payoffs of the game such that  $(\underline{v}^1, \dots, \underline{v}^n) = (0, \dots, 0)$ . I denote by  $IR^*(G, g) = \{g = (g^1, \dots, g^n) \in \mathbb{R}^n : \forall i \in N, g^i > 0\}$  the set of strictly individually rational payoffs. Finally, let  $V^* = \text{co } g(A) \cap IR^*(G, g)$  be the set of feasible and strictly individually rational payoffs.

The aim of this paper is to characterize the networks  $G$  for which a folk theorem holds, that is: each feasible and strictly individually rational payoff is an PBE payoff of the repeated game, for a discount factor close enough to one. In the next section, I introduce and prove the main result.

### 3 Main results

I first study the four-player case. In Section 3.2, I give some examples with more than four players.

#### 3.1 Four-player case

I assume  $n = 4$  throughout this section, and I prove the following theorem.

**Theorem 3.1.** *Assume  $n = 4$ , and that communication is public and local. Then, the following statements are equivalent.*

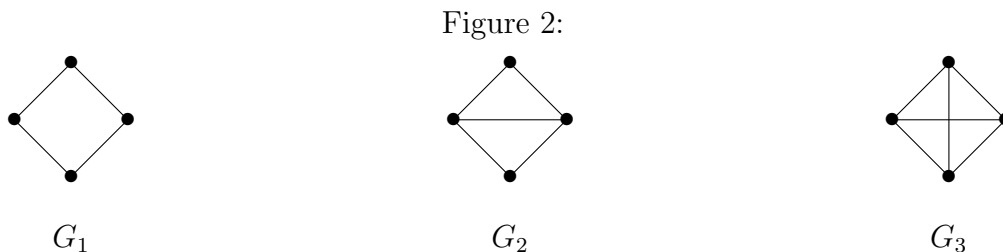
- (1) *The network  $G$  is 2-connected.*
- (2) *For any payoff function  $g$  such that the interior of  $V^*$  is nonempty, for every vector  $v$  in  $V^*$ , there exists  $\bar{\delta} \in (0, 1)$  such that for all  $\delta \in (\bar{\delta}, 1)$ ,  $v$  is a PBE vector payoff of the  $\delta$ -discounted game.*

Renault and Tomala ([20]) show that 2-connectedness is necessary for a Nash folk theorem to hold (for any  $n \geq 3$ ).

*Remark 3.2.* Theorem 3.1 is also true for  $n = 3$  since the unique 2-connected network with three nodes is the complete graph. (Notice that the complete-network case reduces to perfect monitoring.) The 2-player case is also trivial: a folk theorem holds if and only if the graph is connected.

Before proving Theorem 3.1, I set out the following lemma.

**Lemma 3.3.** *Assume  $n = 4$ . Then, the network  $G$  is 2-connected if and only if  $G \in \{G_1, G_2, G_3\}$  where:*



*Proof.* First, it is obvious that  $G_1$ ,  $G_2$  and  $G_3$  are 2-connected. Assume now  $n = 4$ , let  $N = \{1, 2, 3, 4\}$ . Furthermore, suppose that  $G$  is 2-connected, hence each player has at least

two neighbors. Two cases are possible: either (i) each player has exactly two neighbors, or (ii) there exists a player who has three neighbors.

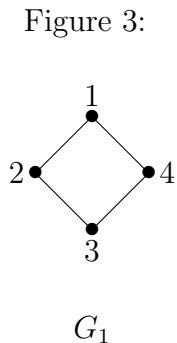
(i) Consider the first case in which each player has two neighbors. Without loss of generality, let  $\mathcal{N}(1) = \{2, 4\}$ . It is neither possible to have  $2 \in \mathcal{N}(4)$  nor  $4 \in \mathcal{N}(2)$ , since it would imply that player 3 has no neighbor. Hence,  $\mathcal{N}(3) = \{2, 4\}$  and the unique possible network is  $G_1$ .

(ii) Assume now that there exists a player who has three neighbors. Notice that  $\sum_{i \in N} \#\mathcal{N}(i) = 2 \times \#E$ , where  $E$  is the set of edges in  $G$ . Hence, the total number of neighbors in  $G$  is even. Therefore, either all players have three neighbors and  $G$  is complete (network  $G_3$ ); or, exactly two players have three neighbors, say players 1 and 3. In the latter case, it must be the case that:  $\mathcal{N}(1) = \{2, 3, 4\}$  and  $\mathcal{N}(3) = \{2, 3, 4\}$ . The unique possible network is then  $G_2$ .  $\square$

I now prove Theorem 3.1. As mentioned before, I only have to prove that (1) implies (2) in Theorem 3.1. Assume that the network  $G$  is 2-connected. By Lemma 3.3, I need to prove that a folk theorem holds for the networks  $G_1$ ,  $G_2$  and  $G_3$ . The case of  $G_3$  reduces to perfect monitoring, hence the proofs of Fudenberg and Maskin ([8, 9]) apply (and no communication is required). In the next sections, I prove that a folk theorem holds for the networks  $G_1$  (Section 3.1.1) and  $G_2$  (Section 3.1.2).

### 3.1.1 Each player has exactly two neighbors

Throughout this section, assume  $n = 4$ , and that communication is public and local. Furthermore, suppose that the game is played on the following network ( $G_1$ ):



Now take a payoff function  $g$  such that  $\text{int } V^*$  is non-empty, and a vector  $v = (v^1, \dots, v^n)$  in  $V^*$ . To prove a folk theorem, I construct a PBE of the repeated game  $(\sigma^*, \phi^*)$  with payoff  $v$  for a discount factor close enough to one. More precisely, I construct a restriction of the PBE strategy to a particular class of private histories; namely, the histories along which only



unilateral deviations from  $(\sigma^*, \phi^*)$ , if any, have taken place. In addition, the construction has the property that, after such histories, the specified play is optimal no matter what beliefs a player holds about their opponents' play, provided that the beliefs are such that: for every player  $i \in N$ , if player  $i$  observes a private history compatible with a history along which no deviation has taken place (respectively along which only unilateral deviations have taken place), then player  $i$  believes that no deviation has taken place (respectively only unilateral deviations have taken place). Plainly, this suffices to ensure optimality. Given that play after other histories (*i.e.*, histories that involve simultaneous deviations) is irrelevant, the partial strategy and beliefs that I define can be completed in any arbitrary fashion. Details are given in Section 3.1.1.

Formally, I denote by  $H_t^i(U|(\sigma^*, \phi^*))$  the set of private histories for player  $i$  such that: either no deviation (from  $(\sigma^*, \phi^*)$ ) has occurred, or only unilateral deviations have taken place. That is to say, for every history in  $H_t^i(U|(\sigma^*, \phi^*))$ , no multilateral deviation has occurred. Similarly, denote by  $H_t(U|(\sigma^*, \phi^*))$  the set of total histories along which only unilateral deviations, if any, have taken place. I define now, for every history in  $H_t(U|(\sigma^*, \phi^*))$ , a strategy profile which can be decomposed into four phases. First, there is a stream of pure action profiles that yields the desired payoff. This is how the game starts off and how it unfolds as long as no player deviates. Second, there is a communication phase (the communication protocol previously described) in case of a deviation, whose purpose is to inform the deviator's neighbors of the deviator's identity. Third, there is a punishment phase, and finally, a reward phase.

### Phase I: equilibrium path

For each player  $i$  in  $N$  and each stage  $t > 0$ , choose  $\bar{a}_t^i \in A^i$  such that

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} g_t^i(\bar{a}_t) = v^i.$$

This is possible when  $\delta \geq 1 - \frac{1}{n}$  (existence is proved by Sorin, Proposition 4 p.151 in [24]). Moreover, Fudenberg and Maskin (1991) prove that for every  $\epsilon > 0$ , there exists  $\delta_\epsilon < 1$  such that for all  $\delta \geq \delta_\epsilon$  and every  $v \in V^*$  such that  $v^i \geq \underline{v}^i$  for all  $i$ , the deterministic sequence of pure actions  $\bar{a}_t$  can be constructed so that the continuation payoffs at each stage are within  $\epsilon$  of  $v$  (Lemma 2 p. 432 in [9]).<sup>1</sup>

Moreover, at every period  $t$ , player  $i$  should send the message  $m_t^i = (1, \emptyset, \alpha_t^i)$  to all his

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<sup>1</sup>If it was not the case, some player would prefer to deviate from  $\bar{a}$ , even if doing so caused his opponents to minmax him thereafter.

neighbors, where  $\alpha_t^i$  is uniformly drawn on  $[0, 1]$ .<sup>2</sup> The number 1 stands for a delay and means “one stage before” (hence stage  $t - 1$ ). The part  $\emptyset$  of player  $i$ ’s message means that he neither deviated nor detected any action deviation at stage  $t - 1$ .<sup>3</sup> In addition, the number  $\alpha_t^i$  can be seen as an encoding key, which is used during the communication phase (phase II).

## Phase II: communication phase

This phase aims at identifying the deviator when a deviation occurs. Formally, the strategy of phase II can be seen as a communication protocol: a specification of how players choose their messages, the number of communication rounds and an output rule for each player. Each player  $i \in N$  starts the communication protocol every time he detects any kind of deviation from  $(\sigma^*, \phi^*)$ . For instance, when in phase I, player  $i$  enters phase II at the end of stage  $t$  either if he observes a neighbor’s deviation (or if he deviates himself in action), or if he receives a message in  $M$ —where  $M$  is the set of messages allowed by the protocol (see the construction below)—different from  $\bar{m}_t^j = (1, \emptyset, \alpha_t^j)$  from some neighbor  $j \in \mathcal{N}(i)$  at stage  $t$  (messages that are not in  $M$  are disregarded). Players may enter phase II at different stages. Indeed, consider the situation in which there is an action deviation of some player  $k$  at stage  $t$ . For instance, if  $k = 1$ , player 1’s neighbors start phase II at the end of stage  $t$ , whereas player 4 does not start phase 2 before the end of stage  $t + 1$ . Moreover, players may start a new communication protocol although a previous one has not ended yet. Therefore, there can be several communication protocols running at the same time.

During phase II, players should stick to the action strategy according to the phase in which the play is. For instance, if players are following the equilibrium path at stage  $t$  when they enter phase II, they should keep playing  $\bar{a}$  when performing the protocol, until a possibly previous protocol ends up and yields to a new phase of the game. This part of the strategy is thereby purely communicative. In what follows, I construct the communication protocol used by player  $i$  in phase II, denoted  $\tilde{\phi}^i$ .

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## COMMUNICATION PROTOCOL

**The message space.** All players communicate using the same infinite set  $M$  of messages. The messages in  $M$  take the following form:

<sup>2</sup>Hence, I assume that the message space  $M^i$  of player  $i$  is uncountable here.

<sup>3</sup>Recall that, at each stage, messages are sent before observing stage payoffs. This assumption is not crucial: with a slight modification, the strategy construction is still valid for the case in which messages are sent after the observation of stage payoffs.

- a delay  $s \in \{1, 5\}$ , where  $T = 5$  is the length of the protocol. The delay  $s$  is interpreted as “ $s$  stages before” and corresponds to the stage of the alleged deviation.
- An element  $x(s)$  in  $\{\emptyset\} \cup N$  which represents the name of the alleged (action) deviator “ $s$  stages before”. Hence, the element  $(s, x_t(s))$  at stage  $s$  is interpreted as “player  $x_t(s)$  deviated in action at stage  $t - s$ .”
- An encoding key  $\alpha$  drawn uniformly on  $[0, 1]$  (which implies that  $M$  is infinite).
- An element of  $F(s) = \emptyset \cup \{(s', x, \alpha^x) : s' \in \{0, \dots, s - 1\}, x \in N, \alpha^x \in [0, 1]\}$ . The interpretation of  $F(s) = (s', x, \alpha^x)$  sent at stage  $t$  is: “player  $x$  deviated in communication at stage  $t - s'$  when he transmitted information about what happened at stage  $t - s$ , and the encoding key sent by player  $x$  at stage  $t - s'$  was  $\alpha^x$ .” There are possibly several such messages since several players might deviate in communication at consecutive stages, but the number of such messages is bounded by two (see the construction of the strategy below). The message  $\emptyset$  stands for the case in which no communication deviation is detected.

With the previous notations, the set  $M$  is the following:

$$M = \left\{ (s, x(s), \alpha, F(s)) : s \in \{1, \dots, 5\}, \alpha \in [0, 1] \right\}.$$

If there are several protocols running at the same time, then each player’s message is a concatenation of each message in  $M$  corresponding to each deviation: different delays correspond to different stages, thus to different protocols. In what follows, I denote by  $x_t^i(s)$  (respectively  $\alpha_t^i$ ,  $F_t^i(s)$ ) the element  $x(s)$  (respectively the number  $\alpha \in [0, 1]$ , the element  $F(s)$ ) sent by player  $i$  at stage  $t$ .

**The strategy of player  $i$ .** When he performs the protocol, player  $i$  sends some messages as a function of his observations, as follows:

- (i) If player  $i$  detects an action deviation of some neighbor  $k \in \mathcal{N}(i)$  at stage  $t$ , then for each  $s \in \{1, \dots, 5\}$ , player  $i$ ’s message at stage  $t + s$  contains  $(s, x_{t+s}^i(s) = k, \alpha_{t+s}^i)$ , where  $\alpha_{t+s}^i$  is drawn uniformly on  $[0, 1]$ .<sup>4</sup> Moreover,  $F_{t+1}^i(1) = F_{t+2}^i(2) = \emptyset$ . Three cases are then possible:

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<sup>4</sup>Notice that player  $i$ ’s message is not complete here, since  $F_{t+s}^i(s)$  is not specified yet. Moreover, several communication protocols might be running at the same time.

- If player  $i$  receives  $F_{t+2}^k(2) = (1, k^1, \alpha_{t+1}^{k^1})$  at stage  $t + 2$  from player  $k$  (recall that player  $k$  is the stage  $t$ 's deviator), where  $k^1 \in \mathcal{N}(k)$  and  $k^1 \neq i$ , then player  $i$ 's messages at stages  $t + 3$ ,  $t + 4$  and  $t + 5$  contain  $F_{t+3}^i(3) = (2, k^1, \alpha_{t+1}^{k^1})$ ,  $F_{t+4}^i(4) = (3, k^1, \alpha_{t+1}^{k^1})$  and  $F_{t+5}^i(5) = (4, k^1, \alpha_{t+1}^{k^1})$  respectively. In other words, player  $i$  transmits the stage  $t + 1$ 's encoding key of player  $k^1$  that he received from player  $k$ , after having updating the delay corresponding to stage  $t + 1$ .
- If player  $i$  receives  $F_{t+2}^k(2) = (1, i, \alpha_{t+1}^i)$  from player  $k$ , and if  $F_{t+4}^k(4) = (1, k^1, \alpha_{t+3}^{k^1})$  at stage  $t + 4$ , with  $k^1 \in \mathcal{N}(k)$  and  $k^1 \neq i$ , then  $F_{t+3}^i(3) = F_{t+4}^i(4) = \emptyset$  and  $F_{t+5}^i(5) = (2, k^1, \alpha_{t+3}^{k^1})$ .
- Otherwise,  $F_{t+s}^i(s) = \emptyset$  for each  $s \in 1, \dots, 5$ .

(ii) If player  $i$  deviates in action at stage  $t$ , then for each  $s \in \{1, \dots, 5\}$ , player  $i$ 's message at stage  $t + s$  contains  $(s, x_{t+s}(s) = i, \alpha_{t+s}^i)$ , where  $\alpha_{t+s}^i$  is drawn uniformly on  $[0, 1]$ . Moreover,  $F_{t+1}^i(1) = \emptyset$ , and:

- If one of his neighbors, say  $i^1$ , does not report stage  $t$ 's deviation at stage  $t + 1$ , *i.e.* either  $m_{t+1}^{i^1} \notin M$  or  $x_{t+1}^{i^1}(1) \neq i$ , then  $F_{t+2}^i(2) = (1, i^1, \alpha_{t+1}^{i^1})$  (recall that player  $i$  receives the value  $\alpha_{t+1}^{i^1}$  at stage  $t + 1$ ) and  $F_{t+3}^i(3) = \emptyset$ . In addition:
  - If player  $i$ 's neighbor other than  $i^1$ , denoted  $i^2$ , does not report  $F_{t+2}^i(2)$  correctly at stage  $t + 3$ , *i.e.* either  $m_{t+3}^{i^2} \notin M$  or  $F_{t+3}^{i^2}(3) \neq (2, i^1, \alpha_{t+1}^{i^1})$ , then  $F_{t+4}^i(4) = (1, i^2, \alpha_{t+3}^{i^2})$  and  $F_{t+5}^i(5) = \emptyset$ .
  - Otherwise,  $F_{t+4}^i(4) = \emptyset$  and  $F_{t+5}^i(5) = \emptyset$ .
- Otherwise,  $F_{t+2}^i(2) = F_{t+3}^i(3) = F_{t+4}^i(4) = F_{t+5}^i(5) = \emptyset$ .

(iii) Otherwise, *i.e.* if player  $i$  does not deviate nor does detect any action deviation at stage  $t$ , then for each  $s \in \{1, \dots, 5\}$ , player  $i$ 's message at stage  $t + s$  contains  $(s, x_{t+s}^i = \emptyset, \alpha_{t+s}^i)$ , where  $\alpha_{t+s}^i$  is drawn uniformly on  $[0, 1]$ . Moreover,  $F_{t+1}^i(1) = \emptyset$ , and:

- If there is one of his neighbors, say  $i^1$ , such that  $m_{t+1}^{i^1} \in M$  and  $x_{t+1}^{i^1}(1) = i$  (*i.e.* player  $i^1$  lies about player  $i$  deviating in action at stage  $t$ ), then  $F_{t+2}^i(2) = (1, i^1, \alpha_{t+1}^{i^1})$  (recall that player  $i$  receives the value  $\alpha_{t+1}^{i^1}$  at stage  $t + 1$ ) and  $F_{t+3}^i(3) = \emptyset$ . In addition:
  - If player  $i$ 's neighbor other than  $i^1$ , denoted  $i^2$ , does not report  $F_{t+2}^i(2)$  correctly at stage  $t + 3$ , *i.e.* either  $m_{t+3}^{i^2} \notin M$  or  $F_{t+3}^{i^2}(3) \neq (2, i^1, \alpha_{t+1}^{i^1})$ , then  $F_{t+4}^i(4) = (1, i^2, \alpha_{t+3}^{i^2})$  and  $F_{t+5}^i(5) = \emptyset$ .

- Otherwise,  $F_{t+4}^i(4) = \emptyset$  and  $F_{t+5}^i(5) = \emptyset$ .
- Otherwise,  $F_{t+2}^i(2) = F_{t+3}^i(3) = F_{t+4}^i(4) = F_{t+5}^i(5) = \emptyset$ .

**The output rule.** Let  $\mathcal{N}(i) = \{i^1, i^2\}$  and  $k \in \mathcal{N}(i^1) \cap \mathcal{N}(i^2) \setminus \{i\}$ . At each stage  $t$ , player  $i$  computes his set of suspected players (regarding stage  $t - 5$ ), denoted  $X_t^i$ , as follows:

- if player  $i$  deviates in action at stage  $t - 5$ , then  $X_t^i = \{i\}$ ;
- if player  $i^1$  (respectively player  $i^2$ ) deviates in action at stage  $t - 5$ , then  $X_t^i = \{i^1\}$  (respectively  $X_t^i = \{i^2\}$ );
- otherwise:
  - if  $m_{t-5+s}^{i^1} \in M$ ,  $m_{t-5+s}^{i^2} \in M$  and  $x_{t-5+s}^{i^1}(s) = x_{t-5+s}^{i^2}(s) = k$  for some stage  $t - 5 + s$  with  $s \in \{1, \dots, 5\}$ , then  $X_t^i = \{k\}$ ;
  - if  $m_{t-5+s}^{i^1} \in M$ ,  $m_{t-5+s}^{i^2} \in M$  and  $x_{t-5+s}^{i^1}(s) = x_{t-5+s}^{i^2}(s) = \emptyset$  for some stage  $t - 5 - s$  with  $s \in \{1, \dots, 5\}$ , then  $X_t^i = \emptyset$ ;
  - if  $m_{t-5+s}^{i^2} \notin M$  and  $m_{t-5+s}^{i^1} \in M$ , for some stage  $t - 5 - s$  with  $s \in \{1, \dots, 5\}$ , then  $X_t^i = \{x_{t-5+s}^{i^1}(s)\}$  (and symmetrically when exchanging the roles of player  $i^1$  and  $i^2$ );
  - otherwise:
    - \* if  $m_{t-2}^{i^1} \in M$  with  $F_{t-2}^{i^1}(3) = (2, i^2, \alpha_{t-4}^{i^2})$  (where  $\alpha_{t-4}^{i^2}$  is the correct value of player  $i^2$ 's encoding key received by player  $i$  at stage  $t - 4$ ), and if  $m_{t-4}^{i^1} \in M$ , then  $X_t^i = \{x_{t-4}^{i^1}(s)\}$  (and symmetrically when exchanging the roles of player  $i^1$  and  $i^2$ );
    - \* if  $m_{t-2}^{i^1} \in M$  with  $F_t^{i^1}(5) = (2, i^2, \alpha_{t-2}^{i^2})$  (where  $\alpha_{t-2}^{i^2}$  is the correct value of player  $i^2$ 's encoding key received by player  $i$  at stage  $t - 2$ ), and if  $m_{t-2}^{i^1} \in M$ , then  $X_t^i = \{x_{t-2}^{i^1}(s)\}$  (and symmetrically when exchanging the roles of player  $i^1$  and  $i^2$ );
    - \* otherwise,  $X_t^i = \emptyset$  (history incompatible with unilateral deviations).

**The number of rounds.** The number of rounds of communication is equal to  $T = 5$ . In the paragraph entitled “The equilibrium property” (page 15), I prove that the following claims hold:

- (1) If there is an action deviation of some player  $k$  at stage  $t$ , then  $X_{t+T}^i = \{k\}$  almost surely for each player  $i \in N$ ;

- (2) If there is no action deviation at stage  $t$  (but possibly a communication deviation), then  $X_{t+T}^i = \emptyset$  almost surely for each player  $i \in N$ .

This ends the description of the communication protocol.

I now describe how transition from phase II to another phase is made. For each player  $i \in N$ :

- if  $X_{t+T}^i = \emptyset$ , then keep playing according to the current action phase and use the corresponding communication strategy;
- if  $X_{t+T}^i = \{k\}$ , then go to phase III in order to minmax player  $k$ ;
- otherwise, play the coordinate of an arbitrary Nash equilibrium of the one-shot game (history incompatible with unilateral deviations).

### Phase III: punishment phase

For each player  $i \in N$ , if  $X_{t+5}^i = \{k\}$  for some player  $k$  in  $N$ , player  $i$  enters a punishment at stage  $t + 6$ , whose goal is to minmax player  $k$ . Each player  $i \neq k$  first plays according to his minmax strategy against  $k$ , denoted  $(P^i(k))$ . Denote by  $P(k) = (P^i(k))_{i \in N-k}$  the profile of minmax strategies against player  $k$ . For any strategy  $(\sigma^k, \psi^k)$  of player  $k$ :

$$\begin{aligned} \gamma_\delta^k(\sigma^k, P(k), \psi^k, (\phi^i)_{i \in N-k}) &\leq \sum_{t=1}^{\infty} (1 - \delta) \delta^{t-1} \underline{v}^k \\ &\leq 0, \end{aligned}$$

where  $\phi^i$  stands for any communication strategy of each minmaxing player  $i \in N-k$ . Player  $k$  is compelled to play some pure action  $P^k(k)$  at each stage during his punishment.<sup>5</sup>

In addition, at each stage  $s$  during the punishment phase, each player  $i$  in  $N$ , including player  $k$ , keeps sending messages in  $M$  regarding stage  $s - 1$ . In particular, a player  $j \in N-k$  who chooses an action which is not in the support of his minmax strategy is identified as a deviator. The punishment phase lasts  $T(\delta)$  stages (the length  $T(\delta)$  has to be adapted, see below).

<sup>5</sup>Otherwise, consider the situation in which some player  $i \neq k$  deviates by reporting to player  $k$  the sequence of pure actions of his minmax strategy. If this minmax strategy were mixed, player  $k$  might benefit from this information by randomizing among his pure actions. However, since player  $k$  is restricted to play the same pure action during his punishment, player  $i$ 's deviation has no impact on player  $k$ 's punishment.

#### Phase IV: reward phase

After the punishment phase, hence at stage  $t + 5 + T(\delta)$ , each player enters a reward phase, whose goal is twofold:

- (i) In order to provide each minmaxing player, who is not minmaxed himself, an incentive to play his minmax strategy in phase III, a reward is added to the form of an additional bonus  $\rho > 0$  to his average payoff. If the discount factor is large enough, the potential loss during the punishment is compensated by the future bonus.
- (ii) Moreover, to induce each minmaxing player to draw his pure actions according to the right distribution of his minmax strategy, I add a phase so that his payoff in the continuation game varies with his realized payoff in a way that makes him indifferent between the pure actions in the support of his minmax strategy. As in Fudenberg and Maskin ([8, 9]), it is convenient to require that any feasible and strictly individually rational continuation payoff must be exactly attained. Otherwise, a minmaxing player might not be exactly indifferent between his pure actions in the support of his minmax strategy.

The possibility of providing such rewards relies on the full dimensionality of the payoff set (recall that  $\text{int } V^* \neq \emptyset$ ). In addition, it is convenient to require that all players know the sequences of pure actions played by each minmaxing player during the punishment phase. For that reason, at the first stage of the reward phase (*i.e.* at stage  $t + 5 + T(\delta)$ ), each player  $i \in N$  sends all the sequences of pure actions that his neighbors  $j \in \mathcal{N}(i)$  played (recall that each player observes his neighbors' moves). The action of player  $i$  at this stage is arbitrary. By replacing  $x(s)$  by the sequences of pure actions played by one's neighbors in the communication protocol of phase II, each player knows at the end of stage  $t + 10 + T(\delta)$  all the sequences actually played by all players. Finally, the construction of the reward phase as well as the specification of  $\mu(\delta)$  is the same as in Fudenberg and Maskin ([9]).

#### Specification of the beliefs

A *belief assessment* is a sequence  $\mu = (\mu_t^i)_{t \geq 1, i \in N}$  with  $\mu_t^i : H_t^i \rightarrow \Delta(H_t)$ : given a private history  $h^i$  of player  $i$ ,  $\mu_t^i(h^i)$  is the probability distribution representing the belief that player  $i$  holds on the full history. An *assessment* is an element  $((\sigma, \phi), \mu)$  where  $(\sigma, \phi)$  is a strategy profile and  $\mu$  a belief assessment.

I consider a restricted set of beliefs, which is strictly included in the set of total histories  $H_t$ . I call this set of beliefs  $\mathcal{B} = (\mathcal{B}^i)_{i \in N}$ . Namely, for each player  $i$  in  $N$ , every belief in  $\mathcal{B}^i$  only assigns positive probability to histories that differ from equilibrium play,  $(\sigma^*, \phi^*)$ , in as much as, and to the minimal extent which, their private history dictates that it does.

Formally, for every player  $i$  in  $N$  and every history  $h_t^i \in H_t^i$ , I denote by  $H_t[h_t^i] \subset H_t$  the set of total histories for which the projection on  $H_t^i$  is  $h_t^i$ . A total history  $h_t$  in  $H_t[h_t^i]$  is said to be compatible with private history  $h_t^i$  of player  $i$ . Now, for every player  $i$  in  $N$  and every history  $h_t^i \in H_t^i$ , let  $H_t[h_t^i](U|(\sigma^*, \phi^*)) \subseteq H_t[h_t^i]$  be the set containing all the total histories that are compatible with  $h_t^i$  and included in  $H_t(U|(\sigma^*, \phi^*))$ .<sup>6</sup> Formally, for each player  $i$  in  $N$  and every history  $h_t^i \in H_t^i$ :

$$H_t[h_t^i](U|(\sigma^*, \phi^*)) = H_t[h_t^i] \cap H_t(U|(\sigma^*, \phi^*)).$$

The set of beliefs  $\mathcal{B}^i$  is then the following:

$$\mathcal{B}^i = \left\{ (\mu_t^i)_{t \geq 1} : \forall t \geq 1, \forall h_t^i \in H_t^i, h_t^i \in H_t^i(U|(\sigma^*, \phi^*)) \Rightarrow \text{supp } \mu_t^i(h_t^i) \subseteq H_t[h_t^i](U|(\sigma^*, \phi^*)) \right\},$$

where  $\text{supp}$  stands for the support of  $\mu_t^i(h_t^i)$ . In other words, the beliefs of the players are such that, if they observe a history compatible with either no deviation or unilateral deviations, then they assign probability one to the fact that the total history is in  $H_t(U|(\sigma^*, \phi^*))$  and is compatible with  $h_t^i$ .

In the next section, I show that for every  $\mu \in \mathcal{B}$ ,  $((\sigma^*, \phi^*), \mu)$  is a PBE with payoff  $v^*$ .

### The equilibrium property

In this section I prove the following proposition.

**Proposition 3.4.** *Assume  $n = 4$ , public and local communication, and  $G = G_1$ . The following statements hold.*

- (a) *If player  $k$  deviates in action at stage  $t$  and if at all stages  $t + s$  with  $s \in \{1, \dots, 5\}$ , at least three players follow  $(\sigma^*, \phi^*)$ , then  $X_{t+5}^i = \{k\}$  almost surely.*
- (b) *If there is no action deviation at stage  $t$  (but possibly an action deviation) and if at all stages  $t + s$  with  $s \in \{1, \dots, 5\}$ , at least three players follow  $(\sigma^*, \phi^*)$ , then  $X_{t+5}^i = \emptyset$  almost surely.*

*Proof of Proposition 3.4.* Assume  $n = 4$ , public and local communication, and  $G = G_1$ . Recall that  $N = \{1, 2, 3, 4\}$  and  $\mathcal{N}(1) = \{2, 4\}$ . I prove points (a) and (b).

(a) Assume without loss of generality that player 1 deviates in action at stage  $t$  and that at least three players follow  $(\sigma^*, \phi^*)$  at each stage, *i.e.* only unilateral deviations are allowed

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<sup>6</sup>Recall that  $H_t(U|(\sigma^*, \phi^*))$  is the set of total histories along which only unilateral deviations, if any, have taken place.



at each stage. Players 2 and 4 observe player 1's moves, hence  $X_{t+5}^1 = X_{t+5}^2 = X_{t+5}^4 = \{1\}$ . At stage  $t + 1$ , three cases are possible.

First, if both players 2 and 4 send  $m_{t+1}^2 \in M$  and  $m_{t+1}^4 \in M$ , which both contain  $(1, x(1) = 1)$ , then by construction  $X_{t+5}^3 = \{1\}$ .

Second, if  $m_{t+1}^2 \notin M$ , then player 4 plays according to  $\tilde{\phi}_{t+1}^4$  under unilateral deviations, hence  $m_{t+1}^4 \in M$  and  $x_{t+1}^4(1) = 1$ . Then by construction,  $X_{t+5}^3 = \{1\}$ . (And symmetrically, if  $m_{t+1}^4 \notin M$ , then  $X_{t+5}^3 = \{1\}$ .)

Finally, assume  $m_{t+1}^2 \in M$ ,  $m_{t+1}^4 \in M$  and that  $x_{t+1}^2(1) \neq 1$ .<sup>7</sup> At stage  $t + 2$ :

- (i) either player 1 deviates, hence players 2 and 4 do not deviate and by construction they both send a message in  $M$  containing  $(2, x(2) = 1)$ : it follows that  $X_{t+5}^3 = \{1\}$ .
- (ii) Or player 1 does not deviate and sends  $m_{t+2}^1 = (2, x(2) = 1, \alpha_{t+2}^2, F_{t+2}^1(2) = (1, 2, \alpha_{t+1}^2))$ . At stage  $t + 3$ , several cases are possible:

- either player 4 does not deviate, thus sends a message in  $M$  containing  $F_{t+3}^4(3) = (2, 2, \alpha_{t+1}^2)$ . Moreover, under unilateral deviations (recall that player 2 deviates at stage  $t + 1$ ), the following holds:  $m_{t+1}^4 \in M$  and  $x_{t+1}^4(1) = 1$ . Finally, the probability that player 2 guesses  $\alpha_{t+1}^4$  is zero since  $\alpha_{t+1}^4$  is uniformly drawn on  $[0, 1]$ . As a consequence,  $X_{t+5}^3 = \{1\}$ .
- Or, player 4 deviates at stage  $t + 3$ . Notice first that if  $m_{t+3}^4 \in M$  and  $x_{t+3}^4(1) = 1$ , the previous reasoning apply and  $X_{t+5}^3 = \{1\}$ . Otherwise, player 1 sends at stage  $t + 4$  a message in  $M$  containing  $F_{t+4}^1(4) = (1, 4, \alpha_{t+3}^4)$ . If player 1 deviates, then both players 2 and 4 follow  $\tilde{\phi}$  and as before,  $X_{t+5}^i = \{1\}$ . Assume now that player 1 does not deviate, then at stage  $t + 5$ , players 2 and 4 should send  $F_{t+5}^2(5) = (2, 4, \alpha_{t+3}^4)$  and  $F_{t+5}^4(5) = (4, 2, \alpha_{t+1}^2)$  respectively. Moreover, under unilateral deviations, the following holds:  $m_{t+1}^4 \in M$ ,  $(1, x(1) = 1) \in m_{t+1}^4$ ,  $m_{t+3}^2 \in M$  and  $(3, x(3) = 1) \in M$ . Finally, either player 2 or player 4 deviates, but not both, hence by construction  $X_{t+5}^3 = \{1\}$ .

Consequently,  $X_{t+5}^3 = \{1\}$  almost surely. This concludes the proof of point (a).

(b) Assume now that there is no action deviation at stage  $t$ . Consider, without loss of generality, player 3's output at stage  $t + 5$ . By construction, 2, 3 and 4 are not in  $X_{t+5}^3$

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<sup>7</sup>The same reasoning apply if  $x_{t+1}^4(1) \neq 1$ .

since player 3 neither deviate nor observe any neighbor's deviation at stage  $t$ . Moreover, under unilateral deviations, there is no stage  $t + s$ , with  $s \in \{1, \dots, 5\}$ , such that  $m_{t+s}^2 \in M$ ,  $m_{t+s}^4 \in M$  and  $x_{t+s}^2(s) = x_{t+s}^4(s) = 1$ . Hence, the only possibility for player 3 to output  $X_{t+5}^3 \neq \emptyset$  is that both following conditions are satisfied:

- (1) player 2 (without loss of generality) sends at stage  $t + 1$  (respectively  $t + 3$ ) a message in  $M$  containing  $(1, x(1) = 1)$  (respectively  $(3, x(3) = 1)$ );
- (2) player 2 announces correctly  $\alpha_{t+1}^4$  at stage  $t + 3$  (respectively  $\alpha_{t+3}^4$  at stage  $t + 5$ ).

The probability of guessing  $\alpha_{t+1}^4$  (respectively  $\alpha_{t+3}^4$ ) is zero. Hence,  $X_{t+5}^3 = \{1\}$  is possible only if player 1 deviate by announcing  $\alpha_{t+1}^4$  (respectively  $\alpha_{t+3}^4$ ) at stage  $t + 2$  (respectively  $t + 4$ ). However, if player 1 deviates at stage  $t + 2$  (respectively  $t + 4$ ), then both players 2 and 4 send at stage  $t + 2$  (respectively  $t + 4$ ) a message in  $M$  containing  $(2, x(2) = \emptyset)$  (respectively  $(4, x(4) = \emptyset)$ ). As a consequence,  $X_{t+5}^3 = \emptyset$  almost surely.  $\square$

By replacing  $x(s)$  by the sequences of pure actions played by one's neighbors during the punishment phase, it follows that the proof of Proposition 3.4 also shows that under  $(\sigma^*, \phi^*)$ , all players know the sequences of pure actions played by others during the punishment phase at stage  $t + T(\delta) + 10$ .

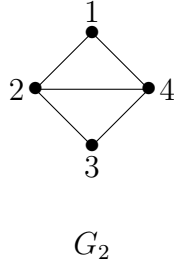
Finally, the proof that the strategy constructed is a PBE is a straightforward application of the proof of Fudenberg and Maskin ([8, 9]).

*Remark 3.5.* Renault and Tomala ([21, 22]) already introduced encoding keys to transmit information through a network. However in their setting, the set of deviators is fixed throughout the repeated interaction, which is not the case here. In my model, the key difficulty is that the set of deviators is not fixed, hence all players may deviate at some stage.

### 3.1.2 Exactly two players have three neighbors

Throughout this section, assume  $n = 4$ , and that communication is public and local. Furthermore, suppose that the game is played on the following network ( $G_2$ ):

Figure 4:



The insights of the previous section 3.1.1 about uniformly distributed keys are not valid anymore, since player 4 knows  $\alpha_t^2$  whether he deviates at stage  $t$  or not. Indeed, communication is public among neighbors, and player 4 receives player 2's announcements.

However, a folk theorem still holds for this network. Indeed it is possible to extend the insights of Ben-Porath and Kahneman ([3]) to prevent both players 2 and 4 from lying.<sup>8</sup> The main idea is to lower by  $\epsilon > 0$  both players 2's and 4's continuation payoffs if their announcements are incompatible at some stage (this is possible because  $\text{int } V^* \neq \emptyset$ ).<sup>9</sup> The argument relies on the fact that communication is costless, hence my construction does not require to minmax a liar who does not deviate in action, rather in communication only. Moreover, incompatible announcements of players 2 and 4 are observed by all players.<sup>10</sup> Hence, all players can start at the same stage the phase in which both players 2's and 4's continuation payoffs are decreased.

Nevertheless, decreasing both players 2's and 4's continuation payoffs creates the following issue: whenever player 2 (without loss of generality) is being minmaxed, he might have an incentive to lie about player 1's deviation in order to reach the state in which both players 2's and 4's continuation payoffs are decreased. For that reason, if players 2's and 4's announcements are incompatible in that player 2 reports player 1's deviation while player 4 doesn't, and if in addition player 2 is being minmaxed, then players start minmaxing player 2 again.

However, it might then be profitable for player 4 not to report player 1's deviations –for instance, player 4 may enjoy minmaxing player 2. To circumvent this issue, I require that whenever player 4 is minmaxed, player 2 gets an additional bonus of  $\rho > 0$  if he reports player 1's deviation.

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<sup>8</sup>Recall that Ben-Porath and Kahneman consider public and global communication.

<sup>9</sup>Notice that the non-equivalent utilities condition introduced by Abreu, Dutta and Smith ([1]) is not sufficient to ensure that this is possible.

<sup>10</sup>Notice that it would not be the case if  $G = G_1$ .

Notice also that players 2's and 4's moves are observed by all players, hence if one of them deviates in action at some stage  $t$ , each player  $i \in N$  identifies the deviator at stage  $t$ .

Finally, to design the reward phase –following each punishment phase–, it is convenient that all players know the sequences of pure actions played by all players during the punishment phase.<sup>11</sup> It is not an issue for players 2 and 4, whose actions are observed by all players. Regarding the sequences of pure actions played by players 1 and 3, I add a subphase at the beginning of the reward phase in which players 2 and 4 announce the sequences played by players 1 and 3. Again, incompatible announcements are punished as before. The action is arbitrary during this subphase.

This strategy is a PBE. The proof of the equilibrium property is a straightforward application of Ben-Porath and Kahneman ([3]), and Fudenberg and Maskin ([8, 9]). Hence, a folk theorem holds for the network  $G_2$  under public and local communication.

*Remark 3.6.* The proof does not require an infinite set of messages. As in [3], the proof of the folk theorem is valid if I impose that  $M$  be finite.

*Remark 3.7.* Regarding the network  $G_1$  of the previous section, incompatible announcements of players 2 and 4 are not observed by all players. Hence, the insights of Ben-Porath and Kahneman do not apply to the network  $G_1$ . Indeed, if  $G = G_1$ , player 2 does not know if player 4's announcement is compatible with his own. One could imagine that players 1 and 4 can transmit this information, but again, the announcements of players 1 and 4 might be incompatible, etc.

## 3.2 More than four players

With more than four players, it is an open question to characterize the networks for which a folk theorem holds under public and local communication. In this section, I show a folk theorem for a specific class of networks (Section 3.2.1), then I introduce an example (Section 3.2.2).

### 3.2.1 A class of networks

**Proposition 3.8.** *Assume  $N = \{x^1, x^2, \dots, x^n\}$ , with  $n \geq 7$ , and that the network  $G$  satisfies the following Condition C:*

$$\forall i, j \in \{1, \dots, n\}, \quad i - j = -3, -2, -1, 1, 2, 3 \pmod{n} \Rightarrow i \in \mathcal{N}(j).$$

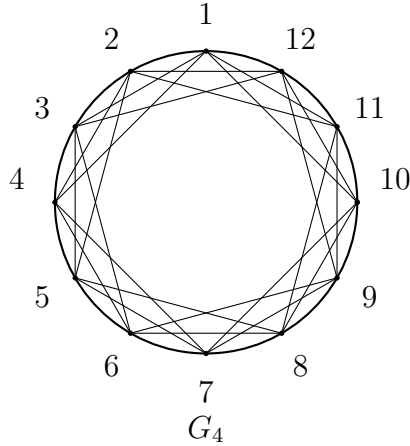
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<sup>11</sup>The minmaxed player, say  $k$ , is compelled to play the same pure action  $P^k$  during his punishment (see footnote 7 page 13).

Then, for any payoff function  $g$  such that the interior of  $V^*$  is nonempty, for every vector  $v$  in  $V^*$ , there exists  $\bar{\delta} \in (0, 1)$  such that for all  $\delta \in (\bar{\delta}, 1)$ ,  $v$  is a PBE vector payoff of the  $\delta$ -discounted game.

The following network satisfies Condition C of Proposition 3.8:

Figure 5:



*Proof of Proposition 3.8.* Assume public et local communication,  $n \geq 7$ , and that the network  $G$  satisfies Condition C of Theorem 3.8. Now take a payoff function  $g$  such that  $\text{int } V^*$  is non-empty, and a vector  $v = (v^1, \dots, v^n)$  in  $V^*$ . To prove a folk theorem, I construct a PBE of the repeated game  $(\sigma^*, \phi^*)$  with payoff  $v$  for a discount factor close enough to one. More precisely, I construct a restriction of the PBE strategy to a particular class of private histories; namely, the histories along which only unilateral deviations from  $(\sigma^*, \phi^*)$ , if any, have taken place. Given that play after other histories (*i.e.*, histories that involve simultaneous deviations) is irrelevant, the partial strategy and beliefs that I define can be completed in any arbitrary fashion.

Formally, I denote by  $H_t^i(U|(\sigma^*, \phi^*))$  the set of private histories for player  $i$  such that: either no deviation (from  $(\sigma^*, \phi^*)$ ) has occurred, or only unilateral deviations have taken place. That is to say, for any history in  $H_t^i(U|(\sigma^*, \phi^*))$ , no multilateral deviation has occurred. Similarly, denote by  $H_t(U|(\sigma^*, \phi^*))$  the set of total histories along which only unilateral deviations, if any, have taken place. I define now, for every history in  $H_t(U|(\sigma^*, \phi^*))$ , a strategy profile which can be decomposed into four phases. First, there is a stream of pure action profiles that yields the desired payoff. This is how the game starts off and how it unfolds as long as no player deviates. Second, there is a communication phase (the communication protocol previously described) in case of a deviation, whose purpose is to inform the deviator's

neighbors of the deviator's identity. Third, there is a punishment phase, and finally, a reward phase.

### Phase I: equilibrium path

During this phase, each player  $i$  in  $N$  should play action  $\bar{a}_t^i$  at stage  $t$  defined as in the proof of Theorem ???. Moreover, at every period  $t$ , player  $i$  should send the message  $m_t^i = (1, \emptyset)$  to all his neighbors. As before,  $(1, \emptyset)$  of player  $i$ 's message means that he did not deviate nor detected any action deviation at stage  $t - 1$ .<sup>12</sup>

### Phase II: communication phase

Fix a player  $i \in N$ . If player  $i$  in  $N$  receives a message different from  $(1, \emptyset)$  at some stage  $t$ , or if player  $i$  observes a neighbor's deviation, then player  $i$  starts a communication protocol, described as follows:

- if a player  $i$  observes an action deviation of a neighbor  $k \in \mathcal{N}(i)$  at stage  $t$ , then he sends the message  $(1, k)$  at stage  $t + 1$ ;
- if a player  $i$  receives from two distinct neighbors, denoted  $i^1$  and  $i^2$ , the messages  $m_t^{i^1} = m_t^{i^2} = (s, k)$  for some  $k \in N$  and  $s \in \{1, T\}$ , where  $T$  is the length of the protocol (see below for the specification of  $T$ ), then  $m_{t+1}^i = (s + 1, k)$ .

As in Section 3.1.1, several communication protocols may be running at the same time: different delays refer to different stages, hence to different protocols.

At each stage  $t$ , player  $i$  computes his set of suspects regarding the alleged deviation stage  $t - s$  for each  $s \in \{0, \dots, T\}$ , denoted  $X_t^i(s)$ , as follows:

- if player  $i$  deviates at stage  $t - s$ , then  $X_t^i(s) = \{i\}$ ;
- if player  $i$  observes the action deviation of some neighbor  $k \in \mathcal{N}(i)$  at stage  $s$ , then  $X_t^i(s) = \{k\}$ ;
- if there exists some  $s' \in \{1, \dots, T\}$  such that there exists two distinct neighbors  $i^1$  and  $i^2$  in  $\mathcal{N}(i)$  such that  $m_{t-s+s'}^{i^1} = m_{t-s+s'}^{i^2} = (s', k)$  for some  $k \in N$ , then  $X_t^i(s) = \{k\}$ .

The number of rounds is  $T = \lfloor n \rfloor - 3$ , where  $\lfloor x \rfloor$  stands for the integer part of  $x$ . For each player  $i \in N$ , the transition rule from phase II to another phase is then the following:

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<sup>12</sup>Recall that, at each stage, messages are sent before observing stage payoffs. This assumption is not crucial: with a slight modification, the strategy construction is still valid for the case in which messages are sent after the observation of stage payoffs.

- if  $X_t^i(T) = \emptyset$ , then keep playing according to the current action phase and use the corresponding communication strategy;
- if  $X_t^i(T) = \{k\}$ , then go to phase III in order to minmax player  $k$ ;
- otherwise, play the coordinate of an arbitrary Nash equilibrium of the one-shot game (history incompatible with unilateral deviations).

**Lemma 3.9.** *Assume public and local communication,  $n \geq 7$ , and that  $G$  satisfies Condition C. The following statements hold.*

- (a) *If player  $k$  deviates in action at stage  $t$ , and if at all stages  $t + s$  with  $s \in \{1, \dots, T\}$ , at least  $n - 1$  players follow  $(\sigma^*, \phi^*)$ , then  $X_{t+T}^i = \{k\}$ .*
- (b) *If there is no action deviation at stage  $t$  (but possibly an action deviation), and if at all stages  $t + s$  with  $s \in \{1, \dots, T\}$ , at least  $n - 1$  players follow  $(\sigma^*, \phi^*)$ , then  $X_{t+T}^i = \emptyset$ .*

*Proof of Lemma 3.9.* Assume public and local communication,  $n \geq 7$ , and that  $G$  satisfies Condition C.

(a) Suppose first that some player  $k$  deviates in action at stage  $t$ , and denote by  $x^2, x^3, x^4, x^{n-2}, x^{n-1}, x^n$  his neighbors. Then  $X_t^i(0) = \{k\}$  for each player  $i \in \{k, x^2, x^3, x^4, x^{n-2}, x^{n-1}, x^n\}$ . Denote by  $x^5$  (respectively  $x^{n-3}$ ) the player such that  $x^5 \in \mathcal{N}(x^1) \cap \mathcal{N}(x^2) \cap \mathcal{N}(x^3)$  and  $x^5 \neq k$  (respectively  $x^{n-3} \in \mathcal{N}(x^{n-2}) \cap \mathcal{N}(x^{n-1}) \cap \mathcal{N}(x^n)$  and  $x^{n-3} \neq k$ ). Players  $x^5$  and  $x^{n-3}$  exist because of Condition C of Proposition 3.8 (if  $n = 7$ , then  $x^5 = x^{n-3}$ ). At stage  $t + 1$ , at least two players among  $x^2, x^3$  and  $x^4$  (respectively among  $x^{n-2}, x^{n-1}$  and  $x^n$ ) send  $(1, k)$ . Hence  $X_{t+1}^{x^5}(1) = \{k\}$ , and symmetrically  $X_{t+1}^{x^{n-3}}(1) = \{k\}$ . By induction, it is easy to see that  $X_{t+T}^i(T) = \{k\}$  for each player in  $N$ .

(b) Assume now that there is no action deviation at stage  $t$ . It is easy to see that under unilateral deviations,  $X_{t+T}^i(T) = \emptyset$  for each player  $i \in N$ . □

### Phase III: punishment phase

For each player  $i \in N$ , if  $X_{t+T}^i = \{k\}$  for some player  $k$  in  $N$ , player  $i$  enters a punishment at stage  $t + T + 1$ , whose goal is to minmax player  $k$ . Each player  $i \neq k$  first plays according to his minmax strategy against  $k$ , denoted  $(P^i(k))$ , defined as in Section 3.1.1. Player  $k$  is compelled to play the same pure action  $P^k(k)$  during his punishment (see footnote 5 page 13).

In addition, at each stage  $s$  during the punishment phase, each player  $i$  in  $N$ , including player  $k$ , keeps sending messages in  $M$  regarding stage  $s - 1$ . In particular, a player  $j \in N^{-k}$  who chooses an action which is not in the support of his minmax strategy is identified as a

deviator. The punishment phase lasts  $T(\delta)$  stages (the length  $T(\delta)$  has to be adapted, see below).

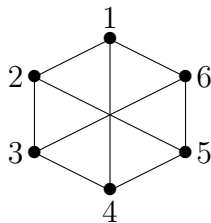
### Phase IV: reward phase

After the punishment phase, hence at stage  $t + T + T(\delta)$ , each player enters a reward phase, whose goal is the same as in Section 3.1.1. The possibility of providing such rewards relies on the full dimensionality of the payoff set (recall that  $\text{int } V^* \neq \emptyset$ ). In addition, it is convenient to require that all players know the sequences of pure actions played by each minmaxing player during the punishment phase. For that reason, at the first stage of the reward phase (*i.e.* at stage  $t + T + T(\delta)$ ), each player  $i \in N$  sends all the sequences of pure actions that his neighbors  $j \in \mathcal{N}(i)$  played (recall that each player observes his neighbors' moves). Then, players use the previous protocol but replace the name of the deviator  $k$  by all the sequences of actions received. By induction, a majority rule at each stage proves that each player knows at the end of stage  $t + 2T + T(\delta)$  all the sequences of pure actions actually played by all players. Players choose  $\bar{a}_t$  at each stage  $t \in \{t + T + T(\delta), \dots, t + 2T + T(\delta) - 1\}$ . Then, players start the reward phase at stage  $t + 2T + T(\delta)$ . The construction of the reward phase as well as the specification of  $\mu(\delta)$  is the same as in Fudenberg and Maskin ([8, 9]). Similar arguments can be found in [12] (Section 5.2.4). The specification of the beliefs is the same as Section 3.1.1. The folk theorem directly follows.  $\square$

### 3.2.2 An example

In the previous section, I show a folk theorem for a specific class of networks (see Figure 3.2.1). However, Condition C of Theorem 3.8 is not necessary for a folk theorem to hold with more than four players. Consider the following network:

Figure 6:



for which Condition C is not satisfied. Nevertheless, it is easy to see that a folk theorem holds for this network under local and public communication. Indeed, the strategy constructed in the proof of Proposition 3.8 is a PBE for this network too. Notice that if player 1 deviates



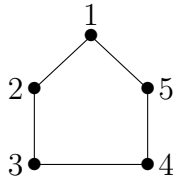
in action at some stage  $t$ , then players 2, 4 and 6 know that the deviator is player 1, and transmit this information at stage  $t + 1$  to both players 3 and 5. A simple majority rule implies that players 3 and 5 know who the deviator is at the end of stage  $t + 1$ .

## 4 Concluding remarks

Under public and local communication, the characterization of the networks  $G$  for which a folk theorem holds remains open if sequential rationality is imposed.

A conjecture is that a folk theorem holds for all circle networks. Indeed, I believe that the strategy in Section 3.1.1 may be extended to more than  $n = 4$  players. However, this is still an open question, and the strategy in Section 3.1.1 needs main modifications to be adapted to more than four players. Indeed, consider the following network:

Figure 7:



The insights of Section 3.1.1 rely on the following observation: if the announcements of players 2 and 5 regarding player 1’s action are incompatible, then one of them is lying, hence player 1 is not, and player 1 thus transmits the liar’s encoding key. The key issue here is that player 3 may not distinguish between the following histories: “the announcements of players 2 and 5 are incompatible”, and “the announcements of players 2 and 5 are compatible, but player 4 is lying about player 5’s announcement”. Hence, if player 3 receives player 2’s encoding key  $\alpha_t^2$  from player 4, player 3 may not know if player 1 deviates at stage  $t + 1$  when reporting player 2’s key, or if player 2 indeed lied at stage  $t$ . Indeed, since player 3 does not know if the announcements of players 2 and 5 are compatible or not, he cannot conclude that player 1 deviates or not when transmitting player 2’s encoding key. As a consequence, the strategy constructed in Section 3.1.1 does not enable to identify the deviator in case of a deviation.

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